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CRANFIELD

Optimum Climb Technique for a Jet Propelled Aircraft*

-by-

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SUMMARY

The present methods for obtaining a climb technique for jet propelled aircraft do not include the effect of kinetic energy variation with height. By introducing the concept of 'energy height' to include the geometric height and the height equivalent of the kinetic energy, a more exact treatment of the optimum technique has been possible. A new method has been suggested for obtaining the energy height climb function.

The energy height optimum climb has been compared with the existing techniques to assess the advantages when specified end conditions are included, the comparison being illustrated by considering a modern fighter project.

Although no great advantage comes from using the energy height optimum climb between given energy heights, a suggestion for zoom climbing at the end of the steady climb has been shown to give a saving of 1 min. 43 secs. in the minimum time to a height of 40,000 ft. for the aircraft considered.

An extension of the energy height method shows how a saving of up to 10 secs. in the time to maximum speed at sea level is possible.

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CONTENTS

1. Introduction

The full tactical operation of a military aircraft requires the knowledge of a climb technique which enables the aircraft to reach the greatest possible height and distance in the shortest possible time, using the minimum amount of fuel, and at the same time reaching the highest possible end speed. A study of the parameters involved shows that there is no general solution to this problem. It was decided, therefore, to divide the study into four main parts, each part covering a specific operational requirement, and capable of further subdivision to meet the detailed tactical employment of the aircraft.

Part I deals with the first operational case 'optimum climb'. This is defined as the case meeting the requirement to reach a certain height and end speed as quickly as possible. It is, in essence, the technique required of a fighter to intercept a bomber which is in visual or radar sight of the operational unit.

Part II covers the 'distance climb' case. It represents the requirements of a fighter to intercept the enemy at a particular radius from a military objective, and so prevent the jettisoning of bombs during the interception phase causing unplanned, but nevertheless serious, damage.

Part III determines the initial climb technique in a long range flight. This is the heavy bomber, reconnaissance, sea patrol, or heavy bomber fighter-escort case.

Part IV deals with the initial climb requirement in a time endurance flight. This case covers all 'patrol' tactical requirements, whether it is the fighter awaiting vectoring instructions, or the close support aircraft awaiting information on targets from the ground troops, or the search aircraft in a coastal or reconnaissance role.

2. Notation

T	Thrust
V	True air speed
V_i	Equivalent air speed = $V\sqrt{\sigma}$
v_c	Rate of climb
V_c, V_{ic}	Optimum speeds for climb
V_Q, V_{iQ}	'Quasi-optimum' speeds
V_{md}, V_{imd}	Speed for minimum drag
H	Geometric height of aircraft above sea level
H_e	Energy height of aircraft $\equiv H + V^2/2g$
$f(H_e, V)$	dH_e/dt
$\phi(H_e, V)$	dt/dH_e
$\chi(H, V)$	dH_e/dt
D	Drag of aircraft
D_h	Drag of aircraft in straight level flight
KD_h	Induced drag of aircraft in straight level flight
W	Weight of aircraft
λ	V/V_{md}
λ_Q	V_Q/V_{md}
τ	$\frac{T}{D_{min}}\left(1 + \frac{V}{T}\frac{\partial T}{\partial V}\right)$
σ	Relative air density
m	V_i/V_{imd}
γ	Inclination of flight path above horizontal
d_{100}	$\frac{K}{\frac{1}{2}\rho_o\, 100^2\, \pi}\left(\frac{W}{b}\right)^2 = 0.0268\, K\left(\frac{W}{b}\right)^2$
K	Induced drag factor in equation $C_D = C_{D_z} + KC_L^2/(\pi A)$
b	Wing-span
D_{100}	Drag at 100 f.p.s. at zero lift.

3. PART I. OPTIMUM CLIMB

3.1. Introduction

The short duration of radar warning, and the high speed and altitude of bomber aircraft has made the minimum time required to climb to any given altitude from ground level the most important characteristic of fighter performance. A few seconds lost can spoil the chances of interception.

In general the optimum technique will be markedly influenced by the end conditions. Ideally, an academic survey should determine the shortest possible time to pass from any one combination of height and speed to any other, and the flight technique to be adopted during the transition between the two states. Such a generalisation would almost certainly prove too unwieldy in practice. The pilot under the high stress and emotion of an operational duty cannot be expected to have the nicety of mathematical appreciation of the tranquil scientist. Consequently the end conditions studied in this report have been restricted to those most likely to occur in military operations, and the transition stage between end conditions reduced to the simplest possible terms.

The high rate of fuel consumption of turbo-jet aircraft makes it probable that fighter aircraft will start the interception direct from take-off without a prior period of patrol. This consideration sets the initial conditions for the climb. When the required interception altitude is reached there are two tactical possibilities:

(i) for a head-on or beam attack the speed of the interceptor is unimportant, and the main consideration is that height should be reached in the least possible time so as to cover the case of least warning

(ii) for a stern attack, or any attack in which 'closing-in' or pursuit of the enemy is essential then the desired height should be reached at the greatest possible speed.

These two requirements determine the final end conditions for the most likely tactical employment of the aircraft.

3.2. The Fundamental Equation

The fundamental equation for the longitudinal motion of the aircraft climbing in g at angle γ in still air can be written approximately

$$T - D - W \sin\gamma = \frac{W}{g}\frac{dV}{dt} \quad \text{..........(1)}$$

If we further denote the total drag of the aircraft in straight level flight by D_h and the induced drag in this condition by $K\,D_h$, then for rectilinear flight at the same speed along a path inclined to the horizontal at angle γ the induced drag is $K\,D_h \cos^2\gamma$ and equation (1) may be written:

$$T - \left\{(1-K)\,D_h + K\,D_h \cos^2\gamma\right\} - W\sin\gamma = \frac{W}{g}\,\frac{dV}{dt}$$

i.e. $$T - D_h + K\,D_h \sin^2\gamma - W\sin\gamma = \frac{W}{g}\,\frac{dV}{dt}.$$

But $$\sin\gamma = \frac{1}{V}\,\frac{dH}{dt},$$

therefore the above may be written:

$$T - D_h + K\,D_h\left(\frac{1}{V}\,\frac{dH}{dt}\right)^2 - \frac{W}{V}\,\frac{dH}{dt} = \frac{W}{g}\,\frac{dV}{dt} \quad \text{..........(2)}$$

or $$\frac{W}{V}\,\frac{dH}{dt}\left\{1 + \frac{V}{g}\,\frac{dV}{dH}\right\} = T - D_h + K\,D_h\left(\frac{1}{V}\,\frac{dH}{dt}\right)^2. \quad \text{..(3)}$$

This is the fundamental equation for the climb from which the requisite technique must be deduced.

3.3. Approximations used in the solution of the Fundamental Equation

3.3,1 The variables involved

The variables T, D_h and K in the fundamental equation are functions of air pressure, air temperature, W and V. In a given atmosphere, therefore, they are functions of W, V, and H.

The weight of the aircraft W is a variable during the climb, but as a first approximation we may take W as constant, and use the standard methods of performance reduction

/to determine ...

to determine the effect on the climb technique of weight variation with height.

We may then say:

$$\frac{dH}{dt} = f\left(H, V, \frac{dV}{dH}\right).$$

3.3,2 <u>The term $K D_h \left(\frac{1}{V}\frac{dH}{dt}\right)^2$</u>

The term $K D_h \left(\frac{1}{V}\frac{dH}{dt}\right)^2$ i.e. $K D_h \sin^2 \gamma$ represents the effect of the <u>reduction</u> in induced drag due to inclination of the flight path to the horizontal. This term is small except in zoom climbs, and will be ignored in the determination of the first approximation of the climb technique between specified end conditions. Its effect on a continuous climb for a fighter is shown in a later stage of this report to be less than 0.2 per cent.

The fundamental equation now becomes:

$$\frac{W}{V}\frac{dH}{dt}\left\{1 + \frac{V}{g}\frac{dV}{dH}\right\} = T - D_h .$$

3.3,3 <u>The term in dV/dH</u>

For propellered aircraft the term in dV/dH, which is a measure of the acceleration along the flight path, may be neglected, because the rate of climb is sufficiently small for the acceleration to be neglected.

Thus for propellered aircraft the fundamental equation becomes:

$$\frac{dH}{dt} = v_c = (T - D_h)\frac{V}{W} \quad .... (4)$$

In the case of jet propelled aircraft, however, the speed in climb is so high that speed variations mean large changes in the energy transformed, and the term can no longer be omitted. Calculations in a later part of this report show that for a modern project fighter with 40° sweepback the effect of this term on estimates of the rate of a continuous climb is proportional to V^2, and may be as high as 12 per cent. Simplification of the equation without omitting the acceleration term was achieved by the Germans who introduced the concept of 'energy height', (reference 1).

/3.4. ...

3.4. <u>The concept of 'energy height'</u>

The energy height H_e is defined as the sum of the geometric, or standard altimeter height, and the speed height, which is the height to which the aircraft could climb by virtue of its K.E.

Thus $$H_e = H + \frac{V^2}{2g}$$

From this definition, movement along a curve of constant energy height proceeds ideally without consumption of energy. This means that the speed along the path with which a definite energy height is attained is immaterial because any other speed may be reached at the same energy height without consumption of energy. Calculations done, therefore, with the energy height, instead of the height of flight, need no correction for acceleration in the flight path. This can readily be demonstrated from the equation of energy for the climb along a small element of the flight path between heights H_1 and H_2 (so that T and D may be assumed constant).

We have from equation (1):

$$TV\,\delta t = D_h\,V\,\delta t + W(H_2-H_1) + W\,(V_2^2 - V_1^2)\,/(2g)$$

i.e. $$\delta t = \frac{H_2-H_1 + (V_2^2 - V_1^2)\,/(2g)}{T - D_h} \cdot \frac{W}{V}$$

$$= \frac{H_{e_2} - H_{e_1}}{T - D_h} \cdot \frac{W}{V}$$

But $V(T-D_h)\,/W$ from equation (4) is the rate of climb v_c for flight uncorrected for acceleration.

Therefore $$\delta t = (H_{e_2} - H_{e_1})\,/v_c\,. \qquad \text{..........(6)}$$

The fundamental equation approximated as indicated in this text, viz:

$$\frac{W}{V}\,\frac{dH}{dt}\left\{1 + \frac{V}{g}\,\frac{dV}{dH}\right\} = T - D_h$$

may be written in terms of this new concept:

$$d\,H_e/dt = (T - D_h)\,V/W\,.$$

Since T and D_h are functions of H and V, they

/are also ...

are also functions of H_e and V and so we may write:

$$d\,H_e/dt \;=\; (T - D_h)\,/W \;=\; f(H_e, V). \qquad \ldots\ldots\ldots\ldots(7)$$

Thus for a degree of approximation in which $\sin^2\gamma$ is neglected, the climb performance of an aircraft of given weight depends on two variables.

3.5. Application of the 'energy height' concept to the derivation of optimum climb technique

Consideration of the energy height concept gives a new approach to the solution of the fundamental equation when the climbing speed is so great that the acceleration term must be included. Previous practice has been to base the best climbing speed on 'partial climbs' made at constant equivalent air speed $V_i \equiv V\sqrt{\sigma}$ and mean height H. At each height on the climb the air speed chosen is such that:

$$\partial v_c / \partial V_i = 0 \quad \text{at} \quad H = \text{constant}.$$

v_c is the rate of climb in a partial climb at constant equivalent air speed through a small range of altitude about the height in question. The partial differentiation is made with H constant and the speed chosen is independent of the acceleration along the climb path. The rate of climb by this method is, therefore, overestimated in the ratio $\left(1 + \frac{V}{g}\frac{dV}{dH}\right) : 1$.

On introducing the new concept, 'energy height', the climb equation, including the acceleration term is (equation 7):

$$\frac{dH_e}{dt} = (T - D_h)\,\frac{V}{W} \;=\; f(H_e, V)$$

and therefore, if ${}_1t_2$ is the time required to change from H_{e_1} to H_{e_2} according to a particular technique:

$${}_1t_2 \;=\; \int_{H_{e_1}}^{H_{e_2}} 1/dH_e/dt \;\; dH_e \;=\; \int_{H_{e_1}}^{H_{e_2}} \phi\,(H_e, V)\; dH_e \text{ , say.}$$

The requirement is for the variation of velocity with energy height, viz. $V = \psi(H_e)$, to be so chosen that this integral is a minimum.

/Euler's ...

Euler's condition for a stationary value of this integral reduces to

$$\frac{\partial}{\partial V}\left(\frac{dH_e}{dt}\right)\Bigg|_{H_e \text{ const.}} = 0$$

Thus there is a stationary value of the 'time to climb' integral whenever the partial derivative with respect to airspeed of the rate of change of energy height, with the energy height kept constant, is zero.

It will be noted that the conditions have been determined for a minimum time to energy height, but it is by no means self evident that this is equivalent to minimum time to geometric height. The equivalence of the conditions is proved in Appendix II.

The quantity dH_e/dt is equal to the rate of climb which would be obtained in a partial climb made at constant true airspeed. Thus the optimum climb technique might be derived from partial climbs made at constant true airspeed, with the mean energy height kept constant.

From the considerations of the previous paragraphs three methods have been developed for assessing the optimum climb technique in the transition stage between specified end conditions. In each case the method has been applied to the 1950 Fighter Project designed in the Department of Design of the College of Aeronautics. The specification of the project is detailed in Appendix I.

3.5,1 <u>Method I</u>

This method is attributable to Lush although it was not explicitly used in reports A. and A.E.E./Res/237 and A. and A.E.E./Res/243.[1,2]

In this method numerical values for thrust and drag obtained either by experiment or analytical assessment are used to determine dH_e/dt and thus $\phi(H_e,V)$ at constant H_e. $\phi(H_e,V)$ is then plotted against true air speed V as in Figures 1 and 2. The values of V and $\phi(H_e,V)$ corresponding to $\frac{\partial}{\partial V}\left(\frac{dH_e}{dt}\right)\Big|_{H_e \text{ const.}}$ are then read from the curves and the the curve of $\phi(H_e,V)_{\text{minimum}}$ against H_e plotted as in Figure 5.

/The time ...

The time taken to climb between two energy heights H_{e_1} and H_{e_2} is the area under this curve between the respective ordinates. The values obtained for H_e, V and dH_e/dt are then used to obtain a solution of the more general performance equation (including the $\sin^2 \gamma$ term) by successive approximation.

For comparison, the values of optimum climbing speed and time to height as deduced from the R.Ae.S. data sheet EG 3/1 have been obtained.[3]

3.5,2 Method II

Lush[1] has used an approximate method based on the theory outlined below.

The fundamental equation, neglecting the $\sin^2 \gamma$ term, has been shown to be of the form

$$dH_e/dt = f(H_e,V) = \chi(H,V),$$

and the Euler condition states that the air speed required for optimum climb is such that

$$\partial f/\partial V = 0.$$

Since $\partial f/\partial V$ depends on derivatives of thrust and drag measured on an energy height scale, it is not readily expressible in terms of familiar quantities. The value $\partial \chi/\partial V$ is more easily deduced, as the thrust and drag terms are now expressed in geometric height units.

For the relationship between these two we may proceed as follows:

since H and V are independent variables,

$$\text{and } f(H_e,V) = f(H + V^2/2g, V) = \chi(H,V)$$

it follows that

$$\frac{V}{g}\frac{\partial f}{\partial H_e} + \frac{\partial f}{\partial V} = \frac{\partial \chi}{\partial V}$$

and
$$\frac{\partial f}{\partial H_e} = \frac{\partial \chi}{\partial H} .$$

/Therefore ...

Therefore $\frac{V}{g} \frac{\partial \chi}{\partial H} + \frac{\partial f}{\partial V} = \frac{\partial \chi}{\partial V}$.

Thus the condition $\partial f/\partial V = 0$ is equivalent to

$$\frac{\partial \chi}{\partial V} = \frac{V}{g} \frac{\partial \chi}{\partial H}$$

Since $(V/g)(\partial \chi/\partial H)$ is small and negative (see Figure 11) it follows that the true optimum speed is a little higher than the quasi optimum speed at which $\partial \chi/\partial V$ is zero. Lush claims that the difference is usually about 5 per cent if no compressibility effects are present. Method I gave a value of 6 per cent for the example chosen, at heights up to 20,000 ft., but this increased to $8\frac{1}{2}$ per cent at 28,000 ft. at which height compressibility effects became apparent. This method in its essence is an approximate estimate of the optimum speed made by estimating the quasi optimum speed and adding 5 per cent to it.

An analytical expression for the quasi optimum speed may be deduced as follows:

$\chi = (T - D) \frac{V}{W}$ by equation (7) and the quasi optimum speed is given by the condition $\frac{\partial \chi}{\partial V} = 0$.

If we neglect the $\sin^2 \gamma$ term, we may write:

$$D = D_{100}\, \sigma \left(\frac{V}{100}\right)^2 + \frac{d_{100}}{\sigma \left(\frac{V}{100}\right)^2}$$

i.e. D is a minimum when $D_{100}\, \sigma \left(\frac{V}{100}\right)^2 = \frac{d_{100}}{\sigma \left(\frac{V}{100}\right)^2}$

This occurs when

$$\frac{V_{md}^4}{100^4}\, \sigma^2 = \frac{d_{100}}{D_{100}} ,$$

i.e. when $\frac{V_{imd}}{100} = \left(\frac{d_{100}}{D_{100}}\right)^{1/4}$.

Then $D_{min} = 2\sqrt{D_{100}\, d_{100}}$

/and we ...

and we may write

$$\frac{D}{D_{min}} = \tfrac{1}{2}\left\{\left(\frac{V_i}{V_{imd}}\right)^2 + \left(\frac{V_{imd}}{V_i}\right)^2\right\}$$

$$= \tfrac{1}{2}\left(m^2 + \frac{1}{m^2}\right), \text{ say.}$$

Therefore substitution in the fundamental equation yields:

$$\chi = \frac{V}{W}\left\{T - \tfrac{1}{2} D_{min}\left(m^2 + \frac{1}{m^2}\right)\right\}$$

For jet aircraft climbing at air speeds near the optimum, γ is not large and W/D_{min} is nearly equal to the maximum ratio $(L/D)_{max}$ which is a constant for the aircraft. Hence the above equation may be written:

$$\frac{\chi\sqrt{\sigma}}{V_{imd}}\left(\frac{L}{D}\right)_{max} = m\left\{\frac{T}{D_{min}} - \tfrac{1}{2}\left(m^2 + \frac{1}{m^2}\right)\right\}$$

since $V = m\,V_{imd}\,/\sqrt{\sigma}$.

It follows that the condition $\partial\chi/\partial V = 0$ is equivalent to:

$$0 = \frac{\partial m}{\partial V}\left\{\frac{T}{D_{min}} - \tfrac{1}{2}\left(m^2 + \frac{1}{m^2}\right)\right\} + m\left\{\frac{1}{D_{min}}\frac{\partial T}{\partial V} - \tfrac{1}{2}\left(2m - \frac{2}{m^3}\right)\frac{\partial m}{\partial V}\right\}.$$

Since $\quad \dfrac{\partial m}{\partial V} = \dfrac{\sqrt{\sigma}}{V_{imd}} = \dfrac{m}{V}$,

we obtain

$$0 = \frac{m}{V}\left\{\frac{T}{D_{min}} - \tfrac{1}{2}\left(m^2 + \frac{1}{m^2}\right)\right\} + \frac{m}{D_{min}}\frac{\partial T}{\partial V} - \frac{m^2}{V}\left(m - \frac{1}{m^3}\right)$$

i.e. $$3m^2 - \frac{1}{m^2} = \frac{2}{D_{min}}\left(1 + \frac{V}{T}\frac{\partial T}{\partial V}\right) = 2\tau \quad \text{say.}$$

Hence if λ_Q is the quasi optimum value of m,

$$\lambda_Q = \sqrt{\frac{\tau}{3}}\left(1 + \sqrt{1 + \frac{3}{\tau^2}}\right)^{\frac{1}{2}}$$

since the real and positive solution is the relevant one.

Thus λ_Q is expressible as a function of τ only. This function which is plotted in Figure 12 was used to determine

/the quasi ...

the quasi optimum speeds for the 1950 Fighter Project. An estimate of the true optimum speed was then deduced and the results are contained in Table VI, p. 19.

3.5,3 Method III

It is considered that this method has some advantages over the two previously discussed.

Curves of equal rate of climb v_c, without correction for acceleration (either measured and reduced to unaccelerated flight, or calculated) are plotted in a V, H, H_e network in Figure 9. These curves were obtained from the v_c, V curves drawn in Figure 8.

It has been shown that the time δt to climb between two energy heights H_e, and $H_e + \delta H_e$ is $\delta t = \delta H_e / v_c$. Moreover in Appendix II it has been shown that the conditions for optimum climb between two energy heights are equivalent to the conditions for optimum climb between two geometric heights. Hence the optimum climb path $V = \mathcal{V}(H)$ must occur at the speeds where the curves of constant energy height are tangential to the v_c curves, i.e. where

$$\frac{\partial}{\partial V}\,(v_c)_{H_e} = \frac{\partial}{\partial V}\left[\frac{dH_e}{dt}\right]_{H_e} = 0 .$$

In Figure 9 the optimum climb path is clearly A, B, C, D, - - - - and this is identical with the one obtained by the more laborious process of Method I.

The climbing times are obtained by graphical or numerical integration from the relation

$$t = \int_{H_{e_1}}^{H_{e_2}} d/H_e / v_c ,$$

where v_c is the rate of climb uncorrected for acceleration.

3.6. RESULTS

3.6,1 Method I

The results obtained for the 1950 Fighter Project using Method I for the determination of the optimum climb path

/are ...

are summarised in Table I.

Table I

H ft.	$\frac{dV}{dH}$ f.p.s./f	V f.p.s.	$\frac{V}{g}\frac{dV}{dH}$	ϕ	$\frac{1}{\phi}$	v_c	V_{mph}	H_e
1,000	.0016	674	.0325	.00592	168.91	163.4	460	8,054
5,000	.004	690.8	.0847	.00636	157.23	144.95	471	12,410
10,000	.0072	720.1	.1583	.00715	139.86	120.75	491	18,051
15,000	.0108	762.7	.2536	.00781	128.04	102.14	520	24,032
20,000	.0120	814	.3030	.00875	114.28	87.71	555	30,289
25,000	.0138	870	.3763	.00994	100.60	73.09	593	36,753
30,000	.0157	927	.4632	.01182	84.60	57.82	632	43,343
		Allowing for Compressibility Effects						
30,000	-.004	905.5	-.1125	.01149	87.03	98.06	617.4	42,725
35,000	-.004	885.4	-.1100	.01366	73.21	82.26	604	47,172
40,000	0	881	0	.01750	57.14	57.14	601	52,053

The following relevant curves have been drawn:

Figs. 1 and 2 - Curves of $\phi(H_e,V)$ against V at a series of constant energy heights

Fig. 3 - Curve of $\phi(H_e,V)$ against V at 45,000 ft. energy height with and without allowance for compressibility drag

Fig. 4 - Optimum climb curve V against H in a network of constant energy heights

Fig. 5 - Curve of $\phi_{min}(H_e,V)$ against H_e giving minimum time to energy height.

The optimum climb path starts at 460 m.p.h. at sea level and this corresponds to an energy height value of 7,069 ft. From this datum the time to various energy heights and the corresponding geometric heights is as set out in Table II (neglecting compressibility).

/Table II ...

Table II

TIME TO HEIGHT UNDER OPTIMUM CLIMB CONDITIONS

Energy Height H_e ft.	Geometric Height H ft.	Time Taken secs.
7,069	S.L.	-
10,000	2,778	17.4
15,000	7,300	49.4
20,000	11,600	84.2
25,000	15,863	122.4
30,000	19,784	164.0
35,000	23,650	209.9
40,000	27,449	259.9
45,000	31,200	309.5

The results quoted in Tables I and II were deduced from the fundamental equation

$$\frac{W}{V}\ \frac{dH_e}{dt} = \frac{W/V}{\phi(H_e, V)} = T - D_h + K\ D_h \sin^2 \gamma$$

neglecting the term in $\sin^2 \gamma$. A first approximation to the effect of this term on $\phi(H_e, V)$ is given in Table III.

Table III

VARIATION IN $\phi(H_e, V)$ WHEN THE $\sin^2 \gamma$ TERM IS INCLUDED IN THE FUNDAMENTAL EQUATION

Energy Height H_e	Geometric Height H	Value of $KD_h \sin^2\gamma$	% Variation in ϕ
10,000	2,778	9.747	0.15
20,000	11,600	5.315	0.11
30,000	19,784	2.765	0.07
40,000	27,449	1.276	0.04

Since the variation in ϕ is less than 0.2 per cent,

/it is ...

it is considered that the omission of the $\sin^2 \gamma$ term is justified.

Calculations for the 1950 Fighter Project based on the Royal Aeronautical Society Data Sheet EG 3/1 were done to enable a comparison to be made between the existing technique and the one suggested by the energy height concept. These calculations are summarised in Table IV.

Table IV

OPTIMUM CLIMB PATH FOR 1950 FIGHTER PROJECT

AS DEDUCED FROM R.Ae.S. DATA SHEET EG 3/1

Height H ft.	V_{mph}	V_{fps}	$\frac{dV}{dH}$	$-\frac{V}{g}\frac{dV}{dH}v_{c_1}$	v_{c_1} (Not allowing for acceleration)	v_{c_2} (Allowing for acceleration)	$\frac{1}{v_{c_2}}$	Equivalent Energy Height ft.	Value of ϕ
S.L.	430.4	631.3	.001	- 3.4	174.5	171.1			
5,000	439.4	644.2	.0028	- 8.8	157.5	148.7	.006725	11444	.006306
10,000	451	661.5	.0033	- 9.7	143.3	133.6	.007485	16794	.006979
15,000	463.8	680.2	.0037	-10.1	129.4	119.3	.008382	22184	.007716
20,000	476.7	699	.0041	-11.6	115.5	103.9	.009624	27587	.008619
25,000	491.3	720.6	.0041	- 9.4	102.1	92.7	.010787	33063	.009771
30,000	504.6	740	.0041	- 6.4	88.7	82.3	.01215	38503	.011386
35,000	518	760	.0045	- 7.7	72.7	65.0	.01538	43969	.01265
40,000	535	784.7	.0050	- 5.29	43.4	38.1	.02624	49562	.01546

The Royal Aeronautical Society Data Sheet states that it is unnecessary in practice to correct the optimum climbing speed for acceleration along the flight path. A correction for acceleration is, however, given for the rate of climb, and this has been incorporated in the results of Table IV. The optimum climb speed obtained in this way has been plotted on Figure 4 for comparison with that deduced by Method I. The comparative rates of climb have been plotted in Figure 6 and the reciprocal of the rate of climb plotted against height in Figure 7 to give the times to altitude. These times are summarised in Table V.

/Table V ...

Table V

TIMES TO GEOMETRIC HEIGHT FOR 1950 FIGHTER PROJECT AS DEDUCED FROM R.Ae.S. DATA SHEET EG 3/1

Geometric Height H	Time Taken secs.
Sea Level	-
5,000	31.9
10,000	67.5
15,000	107.1
20,000	156.6
25,000	202.7
30,000	260.3

3.6.2 Method II

The method of calculation for the optimum climb technique using Lush's Approximate Method are detailed in Reference 1.

The results for the present example are summarised in Table VI.

Table VI

OPTIMUM CLIMBING SPEEDS AS DEDUCED BY LUSH'S APPROXIMATE METHOD

Geometric Height H ft.	V as given by energy height method. m.p.h.	V as given by R.Ae.S. Data Sheet m.p.h.	V as given by Lush's approximation m.p.h.
Sea Level	459	430	457
10,000	491	451	483
20,000	555	477	545
30,000 (without compressibility)	632	505	599
30,000 (with compressibility)	617	-	-
40,000 (without compressibility)	730	535	658
40,000 (with compressibility)	601	-	-

/Method III ...

3.6,3 Method III

The results obtained for the optimum climb path by Method III were, within the accuracy of the curve drawing, identical with those obtained by the more lengthy process of Method I.

The following relevant curves have been drawn:

Fig. 8 - Uncorrected rate of climb against speed at a series of geometric heights

Fig. 9 - Constant uncorrected rates of climb against speed in a network of constant energy heights.

3.7. Appreciation of the Separately Deduced Optimum Climb Techniques

The optimum climb techniques discussed in §§ 3.5,1 - 3.5,3 fall into two distinct classes. One may be termed the 'energy height' technique, and the other is the one normally used viz. that given by the Royal Aeronautical Society Data Sheet EG 3/1. Lush's approximate method is an approximation to the energy height technique.

The technique detailed in EG 3/1 gives a lower forward speed and claims a higher rate of climb than does the energy height technique. In effect it states that if $V = \psi(H)$ is a solution of $dH/dt = (W/V)(T-D_h)$ when $\int_{H_1}^{H_2} (dt/dH)\,dH$ is a minimum, then $V = \psi(H)$ is, in practice, a sufficiently close solution, under the same conditions, to the more exact equation:

$$\frac{dH_e}{dt} = \frac{dH}{dt}\left(1 + \frac{V}{g}\frac{dV}{dH}\right) = \frac{W}{V}(T - D_h).$$

This is clearly true when dV/dH is small, i.e. when the acceleration along the flight path is negligible. In the example taken, which is a typical modern fighter project, the acceleration during the climb is too great for this approximation to be warranted, and there is a wide disparity in the two optimum climb functions. This is shown in the curves of Figures 4 and 9.

The rate of climb has been corrected for acceleration by subtracting $\left(\frac{V}{g}\frac{dV}{dH}\right)\frac{dH}{dt}$ from the rate of climb deduced from the simplified equation, and gives results very close to the true values. The true values have been calculated by reducing the optimum climb function $V = \psi(H)$ to the energy height function $\phi(H_e, V) = dt/dH_e$ and subsequent evaluation

/by a ...

by a step by step numerical process. The estimated and true values for rate of climb are shown in Table VII and are illustrated in Figures 6 and 7.

Table VII

Geometric Height	Rate of Climb as estimated by EG 3/1 f.s.	True rate of climb f.s.
5,000	148.7	150.2
10,000	133.6	134.2
15,000	119.3	120.2
20,000	103.9	105.4
25,000	92.7	93.7
30,000	82.3	81.9

The Royal Aeronautical Society technique, therefore, gives a higher rate of climb and a lower forward speed than the energy height technique. The energy height technique relies on relatively high kinetic energy being acquired at low altitudes where the maximum acceleration is attainable, for conversion into potential energy (i.e. height) at higher altitudes. Longer time is required for the acceleration up to the higher initial climb speed at sea level, and a rather greater time is spent at the lower altitudes, but the excess speed acquired can quickly be converted into height, and the advantages or otherwise of this new technique can only be decided on the overall assessment of the steady climb combined with the end conditions.

To enable a direct comparison to be made, the optimum climb path given by the method of EG 3/1 has been expressed in the terms of the energy height concept and plotted on Figure 5. The values of $\emptyset$ obtained are progressively higher as the energy height increases, showing that this technique gives an increase in the time required to energy height. Therefore, when the end conditions are considered, and kinetic energy can be translated into potential energy, the energy height optimum climb technique must be of greater advantage. The extent of this advantage can only be assessed after a detailed study of the end conditions.

/3.8. ...

3.8. The End Conditions

The transition period of steady climb between the end conditions has been resolved into two techniques. In one a particular geometric height is reached in a shorter time, but at a lower flight speed. It has been shown separately that if the kinetic energy of the aircraft can be rapidly converted into its height equivalent, then the shortest time to a particular height and speed is obtained by using the technique deduced from the energy height concept. In this technique, during a steady climb, a particular geometric height is reached in a longer time, but at a greater flight speed, and the kinetic energy in hand more than compensates for the longer time taken.

Kinetic and potential energy are convertible into height and speed equivalents by 'zoom' climbing and diving. The rate of conversion, however, is dependent on the permissible manoeuvrability of the aircraft. For jet propelled aircraft travelling at high speed, manoeuvrability factors are low. Even at sea level the maximum permissible acceleration is likely to be of the order of 4 g, and at 40,000 ft. geometric height this figure would be of the order of 1.5 g. Near the ceiling, zoom climbing or diving could not be effected. The advantage of using the energy height optimum will depend, therefore, on the margin of kinetic energy gained, and the rapidity with which this can be converted into its geometric height equivalent with the practical limitations on manoeuvrability imposed.

The shortestpossible time to height is of vital importance only to the interceptor fighter. An enemy is unlikely to limit his attack to those heights at which the interceptor jet fighter is most efficient, and consequently the end conditions must be studied at all heights. The tactical employment of the fighter, however, demands, in general, only two possible speeds at the interception height. In a head on attack the minimum control speed is sufficient. For an attack which involves pursuit of the enemy the maximum speed will be required. A head on attack is an unlikely tactical operation with conventional armament, but may be possible with air to air missiles having radar, acoustic, or infra red homing devices. Moreover it may be the only form of attack available to the defence. This will occur when the closing speed of fighter and bomber is small, and insufficient warning of attack is given.

In the work which follows an attempt has been made to assess quantitatively the advantages of using the energy height

/optimum ...

optimum climb technique. The extent to which this advantage can be used in meeting the tactical requirements has been studied; and finally, the optimum climb path from the initial conditions to the actual interception has been determined.

3.9. Quantitative Assessment of the Advantage of using the Energy Height Technique

It will be evident at this stage that no spectacular advantage can be gained by using the energy height technique.

In Figure 5 the function $\phi(H_e, V)$ has been plotted against H_e for the optimum climb defined by the Royal Aeronautical Society, and for the climb defined by the energy height concept. The time taken to pass between two energy heights, H_{e_1} and H_{e_2} is.-

$$\int_{H_{e_1}}^{H_{e_2}} \phi(H_e, V)\ dH_e \ .$$

Hence the time saved is represented by the area between the two curves. The total time which can be gained in passing from an energy height of 7069 ft. (which represents the initial conditions for the energy height climb and is equivalent to 460 m.p.h. at sea level) up to an energy height of 45,000 ft., is of the order of 15 secs. This advantage cannot be completely attained in practice, as it assumes that kinetic and potential energies can be converted into one another without time loss, and without any restrictions on manoeuvrability being imposed.

A number of examples have been evaluated for two aircraft, one using the R.Ae.S. climb and the other the energy height climb, and then both reaching the same end conditions. To make direct comparison possible, the end conditions at each height have been taken, firstly as the optimum climb speed for the R.Ae.S. climb, and secondly a speed of 0.94 M, which is near the maximum attainable for the aircraft considered.

The optimum climb curves plotted in a network of constant energy heights as in Figures 4 and 9 may be interpreted as giving the theoretical means of attaining the same end conditions. They are therefore useful for indicating the best manoeuvre for attaining any particular speed and height relation.

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In the first case both aircraft start at 430 m.p.h. at sea level. One climbs in accordance with the curve C Q D (Figure 4) reaching 25,000 ft. and 491 m.p.h. (Q). The other accelerates at sea level from 430 m.p.h. to 460 m.p.h., and then climbs at the greater speeds represented by A P R to P, where the constant energy height line through Q intersects the energy height optimum climb curve. If the extra kinetic energy at P could be converted immediately into potential energy, it would be equivalent to the geometric height difference between P and Q, and would represent a zoom at infinite speed from P to Q. The time difference between the paths CQ and CAPQ represents therefore the maximum <u>theoretical</u> advantage attainable. This can be shown to be of the order of $5\frac{1}{2}$ secs.

In practice the aircraft would climb to a point about 2,000 ft. below P and then zoom the height difference to Q. In the time taken for this zoom, energy would be put into the aircraft proportional to the energy height difference between the start of the zoom, and the energy height of P and Q. The advantage gained is likely to be about half the theoretical obtainable.

In the second case the first aircraft climbs in accordance with the curve CQD to Q, and then accelerates to S in level flight at 25,000 ft. to 0.94 M flight speed. The second aircraft accelerates at sea level from 430 to 460 m.p.h. and then climbs in accordance with the curve APR to R which is the point of intersection of the constant energy height line through S and the energy height optimum climb curve. From reasoning identical with that in the first case the time difference between these paths represents the maximum theoretical advantage attainable. This can be shown to be of the order of $7\frac{1}{2}$ secs.

In practice a dive would be commenced at about 26,000 ft. geometric height corresponding to the point Y on the climb curve. In the time taken for this dive, energy would be put into the aircraft proportional to the energy height difference between Y and R. The advantage gained is likely to be about 5 secs.

Two further examples have been evaluated, namely, from initial conditions to 30,000 ft. and 505 m.p.h., and initial conditions to 30,000 ft. and 648 m.p.h. The theoretical advantages attainable have been shown to be $8\frac{1}{2}$ and 13 secs. respectively.

/It would ...

It would appear that some advantage is to be gained by building up kinetic energy at a low level and then zoom climbing to height. It was necessary to investigate, therefore, whether it would be advantageous to increase speed at sea level up to near the maximum, and then zoom climb to a height and speed on the optimum climb curve. A steady climb would be continued from this position.

In the analysis the aircraft took 65 secs. to accelerate from 460 m.p.h. to 680 m.p.h. at sea level. The higher speed corresponds to an energy height of 15,400 ft. If the increased kinetic energy gained could be converted immediately into its height equivalent, the aircraft would zoom to the point on the optimum climb curve corresponding to 473 m.p.h. and 7,900 ft. geometric height. In practice we should have to allow for the energy input from the engine with a result that a vertical zoom would take the aircraft to 8,750 ft. and a speed of 487 m.p.h. The time taken, allowing 2 secs. loss of advantage at each end for transition from zoom to steady climb conditions, would be 14 secs., giving a total time of 79 secs.

The time taken to reach the same end conditions by proceeding along the optimum climb curve would be 60 secs., representing a saving of 19 secs. for the operation. Thus time is lost in increasing the speed to near the maximum at sea level and then zoom climbing. This result is to be expected from the theory in support of the energy height optimum climb technique. An increase of speed at sea level represents an increase of energy height, and deductions from Euler's condition for the minimum time between two energy heights shows that this is best achieved under the conditions set by the optimum climb curve.

3.10. Optimum Climb to Height when the End Conditions are Included

The end conditions in this investigation have been defined as the minimum control speed and the maximum speed at a stated geometric height.

3.10,1 Optimum climb to geometric height H and the minimum control speed V_{min} (EAS)

It is evident from the foregoing study that the theoretical optimum climb is to accelerate at sea level to the

/evaluated ...

evaluated initial conditions (460 m.p.h. for the 1950 Fighter Project) and then climb steadily following the optimum climb curve to an energy height of $H + V^2_{min}/(2\sigma g)$ finishing with a zoom climb to the geometric height H. In practice the steady climb will end at an energy height $H + V^2_{min}/(2\sigma g) - \delta^2$ where δ^2 depends on the manoeuvrability factor of the aircraft at that height. The difference in energy heights δ^2 between the theoretical optimum and practical will be a measure of the energy put into the aircraft during the zoom.

To illustrate these deductions let the minimum control speed at 40,000 ft. be 180 m.p.h. EAS or 363 m.p.h. TAS. The energy height equivalent of the end conditions is 44,394 ft. This energy height is reached on the optimum climb curve at a geometric height of 31,000 ft. and at a flight speed of 635 m.p.h. TAS. Theoretically a zoom could be done from this geometric height and speed converting the excess kinetic energy into the geometric height difference between 31,000 ft. and 40,000 ft. The total time taken for the climb from the initial conditions to the required energy height is obtained from the curve in Figure 5 as $\int_{7069}^{44,394} \phi(H_e, V)\, dH_e$ and is 304 secs. The time taken to do a steady climb to 40,000 ft. using the R.Ae.S. recommendations would be (from Figure 7) 415 secs. Thus the theoretical advantage attainable is of the order of 1 min. 51 secs.

In practice the zoom will commence at a lower geometric height and take a finite time, but the greater portion of the theoretical advantage can be attained. The loss in theoretical advantage is equal to the time taken for the zoom less the difference in times taken to reach an energy height of 44,394 ft. and the energy height at which the zoom commences. If the zoom commences at 30,000 ft. geometric height instead of the theoretical 31,000 ft., the loss in advantage would be the time for the zoom less 11 secs. Thus even allowing a zoom time of 20 secs., an overall advantage in the time to climb 40,000 ft. of 1 min. 42 secs. would be attained.

By way of illustration, a Meteor will zoom from 35,000 ft. to 40,000 ft. in 9 secs., the speed falling from 520 m.p.h. to 392 m.p.h.

Thus the absolute minimum time to height for the 1950 Fighter Project is achieved in three stages.-

(a) An acceleration at sea level to 460 m.p.h.

(b) A steady climb in accordance with the following table:

H ft.	V_c mph.	V_{i_c} mph.
0	460	460
5,000	471	437
10,000	491	422
15,000	520	413
20,000	555	405
25,000	593	397
30,000*	632	387
30,000+	617	377
35,000+	604	336
40,000+	601	298

* Not allowing for compressibility

\+ Allowing for compressibility

(c) A zoom climb to height. The zoom will commence at an energy height 500 to 1,500 ft. below the energy height value given by the required geometric height, and the minimum flying control speed of the aircraft at that height. The lower figure quoted will apply to lower altitudes, and the greater figure for altitudes in the neighbourhood of 40,000 ft. The figure cannot be determined accurately by analysis, but is readily obtainable from a flight test.

At heights very near the ceiling, the minimum control speed, and the optimum climb speed, are nearly equal and a zoom climb is not possible.

3.10,2 Optimum climb to geometric height H and the maximum speed V_{max}

It is evident that the theoretical optimum climb is to accelerate at sea level to the evaluated initial conditions, and then climb steadily following the optimum climb to an energy height of $H + V^2_{max}/(2g)$ finishing with a zoom dive to the geometric height H.

The optimum climb speed approaches the maximum speed at height and consequently the geometric height of overshoot decreases. At 25,000 ft. geometric height the overshoot is of the order of 1,800 ft. and the theoretical advantage attainable is $7\frac{1}{2}$ secs., whilst at sea level the maximum speed is best

/attainable ...

attainable theoretically by a climb to 7,750 ft. with a possible time saving of 22 secs.

In practice, lack of manoeuvrability of a high speed jet aircraft sets a severe limit on the steepness of the zoom dive. Even with 4 g manoeuvrability, which is only possible at low levels, the following 'pull out' radii are required:

V m.p.h.	Radius of Pull-out ft.
684	10,300
497	5,480
311	2,150
186	768

At 30,000 ft. the factor is likely to be down to 2 g and at 40,000 ft., 1.5 g.

In practice the theoretical advantage is not fully attainable. The dive must be commenced at a lower energy height than the theoretical and approximately half the theoretical advantage can be achieved.

Calculations show that a saving of 10 secs. in the time to a speed of $M = 0.9$ at sea level can be gained by a climb to 5,000 ft. geometric height and a zoom dive of $22^{\circ}50'$ from the horizontal. This dive is illustrated in Figure 10. At 25,000 ft. the overshoot in practice will be of the order of 1,000 ft. and the time saved 4 secs. The magnitude of the overshoot will best be determined by flight testing when the maximum diving angle is known.

3.11 Method of Obtaining the Optimum Climb Relationship for any Particular Aircraft

If the thrust and drag figures for the aircraft are known, the optimum climb curve can be obtained from the methods of analysis described.

In flight it has been shown that the requisite information is given by partial climbs at constant energy height. Owing to the small time intervals involved this is not likely to prove accurate. A better method would be to determine the accelerations in straight and level flight at a series of

/geometric ...

geometric heights. This would give values for $(T - D_h)$ from which $\phi(H_e, V)$ could be determined by the methods of analysis previously quoted.

3.12. Conclusions

1. The optimum climb curve obtained by using the energy height concept gives the most rapid climb between specified end conditions.

2. The gain in time of this climb over that deduced from the R.Ae.S. Data Sheet EG 3/1 between two energy heights is small, being less than 15 secs. between energy heights of 7,000 and 40,000 ft. for the particular aircraft considered.

3. The minimum time to height is achieved by finishing the steady climb with a zoom climb until the aircraft speed reaches the minimum value for effective control. In this climb a saving of 1 min. 41 secs., for the example taken, as compared with a climb following the R.Ae.S. Data Sheet method is achieved in going from 460 m.p.h. at sea level, to minimum effective control speed at 40,000 ft.

4. The minimum time to maximum speed at a height below the service ceiling is achieved by overshooting the geometric height by an amount in theory varying from 7,400 ft. at sea level to 1,800 ft. at 25,000 ft. resulting in an overall time saving of 22 secs. and 7½ secs. respectively as compared for the aircraft considered with the results given by the R.Ae.S. Data Sheet method. In practice, allowing for manoeuvrability limitations, these figures become 5,000 ft. and 1,000 ft., and the time savings 10 secs. and 4 secs. for the example chosen.

5. The optimum climb is best obtained in flight testing by carrying out a series of acceleration measurements in straight and level flight at various geometric heights.

6. The approximate method for determining the optimum climb speed given by Lush[2] is satisfactory at low altitudes but is in error by approximately 7 per cent at heights in the neighbourhood of 40,000 ft.

4. PART II. Distance Climb

4.1. Introduction

In this section of the report the best climb to intercept the enemy at a particular radius from a military objective is investigated.

There are two main aspects to be considered. Firstly, the tactical requirement may be for interception in the shortest possible time regardless of fuel expenditure. This will occur when the range of the fighter is not unduly limited by the amount of fuel carried, or the interception is so critical that the tactical commander is prepared to accept a shorter time of contact with the enemy. In this case success in the operation may depend on the number of fighters engaged. Secondly, the requirement may be for interception in the least time, with fuel economy an over-riding consideration, to permit as long a time of contact with the enemy as possible.

Between these two cases there is an infinite number of variations. It is felt, however, that a detailed study of these particular end conditions will give a guide to the best technique to be adopted in any practical case.

4.2. Minimum time to height at a particular radius when fuel economy is not essential

The horizontal distance covered in the optimum climb from the initial conditions of 460 m.p.h. at sea level to 40,000 ft. is of the order of 25 miles. This distance is proportionately less for lower heights of interception. It is assumed that the radius at which contact is to be made is appreciably greater than these distances.

Maximum speed ($M = 0.96$) at sea level is 731 m.p.h. and this figure decreases with height, becoming 634 m.p.h. at 40,000 ft. Thus a 15 per cent saving in time from this consideration alone is achieved if the interception radius can be covered at sea level. The fuel consumption at maximum speed at sea level, however, is more than 3 times greater than that at 40,000 ft.

Calculations have shown that the maximum speed at sea level can best be attained by an optimum climb from the initial conditions (460 m.p.h.) until an energy height is reached which

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is slightly less than that corresponding to sea level and maximum speed, namely $0 + (1072)^2/(2g)$ ft. The aircraft is then dived at an angle not exceeding about 23° to sea level.

In the example chosen a theoretical advantage of 32 secs. is attainable in reaching a speed of 0.9 M, and a 10 sec. practical advantage is probable. The theoretical energy height is 15,400 ft. corresponding to a geometric height of 7,750 ft. and 483 m.p.h. In the practical case the 10 sec. advantage is gained by a climb to a geometric height of 5,000 ft. followed by a 23° (measured from the horizontal) dive. For a final speed of 0.96 M the theoretical advantage would be approximately 45 secs. and the bulk of this could be gained in an interception at considerable radius. The aircraft would be climbed initially to a geometric height of 10,000 ft. and a long dive would follow.

The climb to height would start at about 20 miles from the required radius of interception. The excess kinetic energy would be converted into its height equivalent by a zoom climb to a point on the optimum climb curve. This zoom would take the aircraft to about 12,000 ft. A steady climb would follow in accordance with the relation previously established. The steady climb would then be followed by a zoom to the minimum control speed for the minimum overall time to be achieved. If, however, it is required to reach the interception height at the maximum speed, the steady climb will be continued beyond the interception height, and the aircraft dived to the maximum speed. These two procedures have been detailed in Part I of this report.

Two further advantages of covering part of the distance to the interception radius at sea level are:

(a) the manoeuvrability factor being higher, alterations of direction due to changes of vectoring instructions can more readily be carried out

(b) the final climb to height will be achieved in less time owing to the decrease in aircraft weight corresponding to the consumption of fuel.

4.3. Best time to height at a particular radius when fuel economy is a prime consideration

If fuel economy is a requirement up to the time of contact with the enemy, then attainment of maximum speed at sea

/level is ...

level is precluded. It is necessary to attain height as quickly as possible. The aircraft should be climbed on the optimum climb to an energy height $H + V_{max}^2/(2g)$, and then dived to reach maximum speed at geometric height H. Flight is maintained at this speed up to the required radius.

A further saving in fuel could be effected, at the expense of time, by zoom climbing from the optimum climb to the required height, and then accelerating from the minimum control speed to the maximum speed in straight level flight.

4.4. Conclusions

The minimum time to height at a particular radius is, therefore, achieved in the following stages:

(a) The maximum speed at sea level (0.96 M) is reached by following the optimum climb to approximately 10,000 ft. and then diving at a small angle to sea level.

(b) Maximum speed is maintained at sea level until within twenty miles of the required interception radius, and then a zoom climb of approximately 10,000 ft. to a point on the optimum steady climb curve.

(c) The optimum climb is then followed until within zooming reach of the required height, the height being attained at the minimum control speed. For interception at 40,000 ft. the aircraft is within zooming reach at 30,000 ft.

The minimum time to height at a particular radius and maximum speed is obtained by carrying out procedures (a) and (b) above, and then continuing the steady climb on the optimum climb curve to an energy height slightly less than $H + V_{max}^2/(2g)$. The aircraft is then dived as steeply as possible to reach maximum speed at geometric height H.

When fuel economy is to be considered, the best procedure is as follows:

(a) A steady climb following the optimum is made to an energy height $H + V_{max}^2/(2g)$ followed by a dive to reach maximum speed at geometric height H.

(b) Maximum speed at geometric height H is maintained until the required radius is reached.

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An even greater fuel economy, at the expense of time, can be effected by zoom climbing from the optimum climb curve to reach the geometric height H at the minimum control speed, and then accelerating in straight and level flight to the required radius.

5. PART III. Initial Climb for Long Range Flight

The Specific Air Range (S.A.R.) is given by:

$$S.A.R. \ = \ \frac{60\ V}{88} \ x \ \frac{8.1}{C \ x \ Drag} \quad \text{nautical m.p.g.},$$

where C = Specific fuel consumption, lb./hr./lb. thrust, and assuming a fuel density of 8.1 lb./gal.

The maximum specific air range is thus obtained at the value of V_i for which

$$\frac{Drag}{V} = \left\{ D_{100} \left(\frac{V_i}{100} \right)^2 + \frac{d_{100}}{(V_i/100)^2} \right\} \frac{\sqrt{\sigma}}{V_i}$$

is a minimum if C is assumed independent of speed. This occurs when $m + 1/m^3$ is a minimum, where $m = V_i / V_{imd}$,

i.e. when $m = 3^{1/4} = 1.31$.

$$\text{Hence maximum S.A.R.} \ = \ \frac{317}{D_{100}^{3/4} \ d_{100}^{1/4} \ C \sqrt{\sigma}} \quad \text{nautical m.p.g.}$$

C, in fact, decreases with increase in altitude.

Thus in attaining the best specific air range the overriding variable is height, and neither V_i nor W are critical ($d_{100} \propto W^2$).

For the example considered, the specific fuel consumption is at 11,600 r.p.m.

The maximum range, therefore, is attained by climbing to the ceiling following an energy height optimum climb obtained by using thrust figures corresponding to 11,600 r.p.m. Near the ceiling manoeuvrability factors are too low for zooms to be effected. The aircraft will continue to climb slowly with $V_i = 1.31\ V_{imd}$ as W, and therefore d_{100} decreases due to the consumption of fuel.

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6. Part IV. Initial Climb for Time Endurance Flight

The speed for maximum endurance is clearly the minimum drag speed V_{imd}. The specific fuel consumption decreases with height. Consequently increase of height means an increase in the time endurance of an aircraft.

For the example chosen the specific fuel consumption is a minimum at 11,600 r.p.m.

The best climb to height for a time endurance flight is, therefore, to climb at an energy height optimum climb using the thrust giving minimum specific fuel consumption. In the example chosen this is achieved by using 11,600 engine r.p.m. The energy height climb path can be obtained by one of the methods detailed in Part I.

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2. Lush, K.J. The loss in climb performance, relative to the optimum, arising from the use of a practical climb technique. Report No. A. and A.E.E./Res/243, Aug. 1949.

3. Royal Aeronautical Society Performance Data Sheet EG 3/1, October, 1950.

/APPENDIX I ...

APPENDIX I

Specification of 1950 Fighter Project and Assumptions used in the Analysis

Specification

Wing	Gross area	602 sq.ft.
	Net area	536 sq.ft.
	t/c root	0.08
	t/c tip	0.07
	SMC	11.92 ft.
	Transition point	10 per cent
	Position of max. thickness	35 per cent
	Span	50.5 ft.
	Angle of sweepback	40°
Fuselage	Wetted area	1040 sq.ft.
	Maximum diameter	5.75 ft. approx.
	Length	58 ft.
	Transition point	Nose
Tail-plane	Net area	143.5 sq.ft.
	SMC	7.6 ft.
	Position of max. thickness	40 per cent
	t/c	0.07
	Transition point	L.E. (since probably in wing wake)
	Fin and Rudder	
	Net area	56.7 sq.ft.
	SMC	7.2 ft.
	Position of max. thickness	35 per cent
	t/c	0.07
	Transition point	10 per cent

Drag Summary

Wing	32.4 lb.
Tail-plane	8.2 lb.
Fin and rudder	3.6 lb.
Fuselage	22.2 lb.
Roughness- wing	7.1 lb.
fuselage	1.8 lb.
tail	2.8 lb.
Interference - wing/body	1.6 lb.
tail/body	1.0 lb.
Control gaps - tail	3.0 lb.
wing	1.6 lb.
canopy	1.5 lb.
	86.8 lb.
+ 12 per cent leaks, excrescences etc. =	97 lb.

and $C_{D_z} = 0.0135$

/For ...

For drag increase with Mach Number, the following assumptions have been made:-

M	< 0.90	0.91	0.92	0.93	0.94	0.95	0.96
C_{D_z}	0.0135	0.0142	0.0154	0.0175	0.0207	0.0250	0.0305

W is assumed constant at 95 per cent of the take off weight, namely 26,600 lbs.

Engines

Two turbojets of static thrust 30 per cent greater than the Rolls Royce Nene.

Engine setting taken as 12,300 r.p.m.

General

K, the induced drag factor is assumed to be 1.1.

Atmospheric conditions are taken as standard I.C.A.N.

APPENDIX II

Proof that the minimum time to Energy Height leads to the minimum time to Geometric Height for the same end conditions

The time to height may be written in either of the alternative forms:

$$t = \int_{H_{e_1}}^{H_{e_2}} \phi(H_e, V)\, dH_e \qquad \text{..........(A.1)}$$

or

$$t = \int_{H_1}^{H_2} F(H, V)\, dV/dH \qquad \text{..........(A.2)}$$

where

$$H_e = H + V^2/(2g) \qquad \text{..........(A.3)}$$

It follows from §3.5 that Euler's condition for a minimum value of (A.1) is equivalent to

$$\left.\frac{\partial \phi}{\partial V}\right|_{H_e \text{ const.}} = 0 \qquad \text{..........(A.4)}$$

and the condition for (A.2) to be a minimum is

$$\left.\frac{\partial F}{\partial V}\right|_{H, v_h} - \frac{d}{d\,H}\left(\left.\frac{\partial F}{\partial v_h}\right|_{H,V}\right) = 0$$

where $v_h = dV/dH$.

From equation (A.3) $dH_e = (1 + v_h\, V/g)\, dH$

therefore

$$\phi(H_e, V)\,(1 + v_h\, V/g) = F(H, V, v_h) \qquad \text{..........(A.5)}$$

Treating V, H_e and v_h as independent variables we have, from equation (A.5):

$$\left.\frac{\partial F}{\partial V}\right|_{H, v_h} = \left\{\left.\frac{\partial \phi}{\partial V}\right|_{H_e} + \frac{V}{g}\left.\frac{\partial \phi}{\partial H_e}\right|_{V}\right\}\left\{1 + \frac{V}{g}\, v_h\right\} + \frac{v_h\,\phi}{g} \qquad \text{....(A.6)}$$

and

$$\left.\frac{\partial F}{\partial v_h}\right|_{H,V} = \frac{V}{g}\,\phi \qquad \text{..........(A.7)}$$

/Therefore ...

Therefore $\frac{d}{dH}\left(\left.\frac{\partial F}{\partial v_h}\right|_{H,V}\right) = \frac{v_h}{g}\,\phi + \frac{V}{g}\,\frac{d\phi}{dH}$ ....(A.8)

But $\frac{d\phi}{dH} = \left.\frac{\partial\phi}{\partial H_e}\right|_V \left\{1 + \frac{V}{g}\,v_h\right\} + \left.\frac{\partial\phi}{\partial V}\right|_{H_e} v_h$ ....(A.9)

and hence equation (A.8) may be written

$$\frac{d}{dH}\left[\frac{\partial F}{\partial v_h}\right]_{H,V} = \frac{v_h}{g}\,\phi + \frac{V}{g}\,\left.\frac{\partial\phi}{\partial H_e}\right|_V \left\{1 + \frac{V}{g}\,v_h\right\} + \frac{V}{g}\left(\left.\frac{\partial\phi}{\partial V}\right|_{H_e}\right) v_h \qquad \text{..........(A.10)}$$

Subtracting (A.10) from (A.6):

$$\left.\frac{\partial F}{\partial V}\right|_{H,v_h} - \frac{d}{dH}\left(\left.\frac{\partial F}{\partial v_h}\right|_{H,V}\right) = \frac{\partial\phi}{\partial V}_{H_e}$$

Hence the condition $\left.\frac{\partial\phi}{\partial V}\right|_{H_e} = 0$ leads to the

equivalent condition

$$\left.\frac{\partial F}{\partial V}\right|_{H,v_h} - \frac{d}{dH}\left[\frac{\partial F}{\partial v_h}\right]_{H,V} = 0$$

Thus the minimum time to energy height leads to the minimum time to geometric height allowing for the same end conditions.

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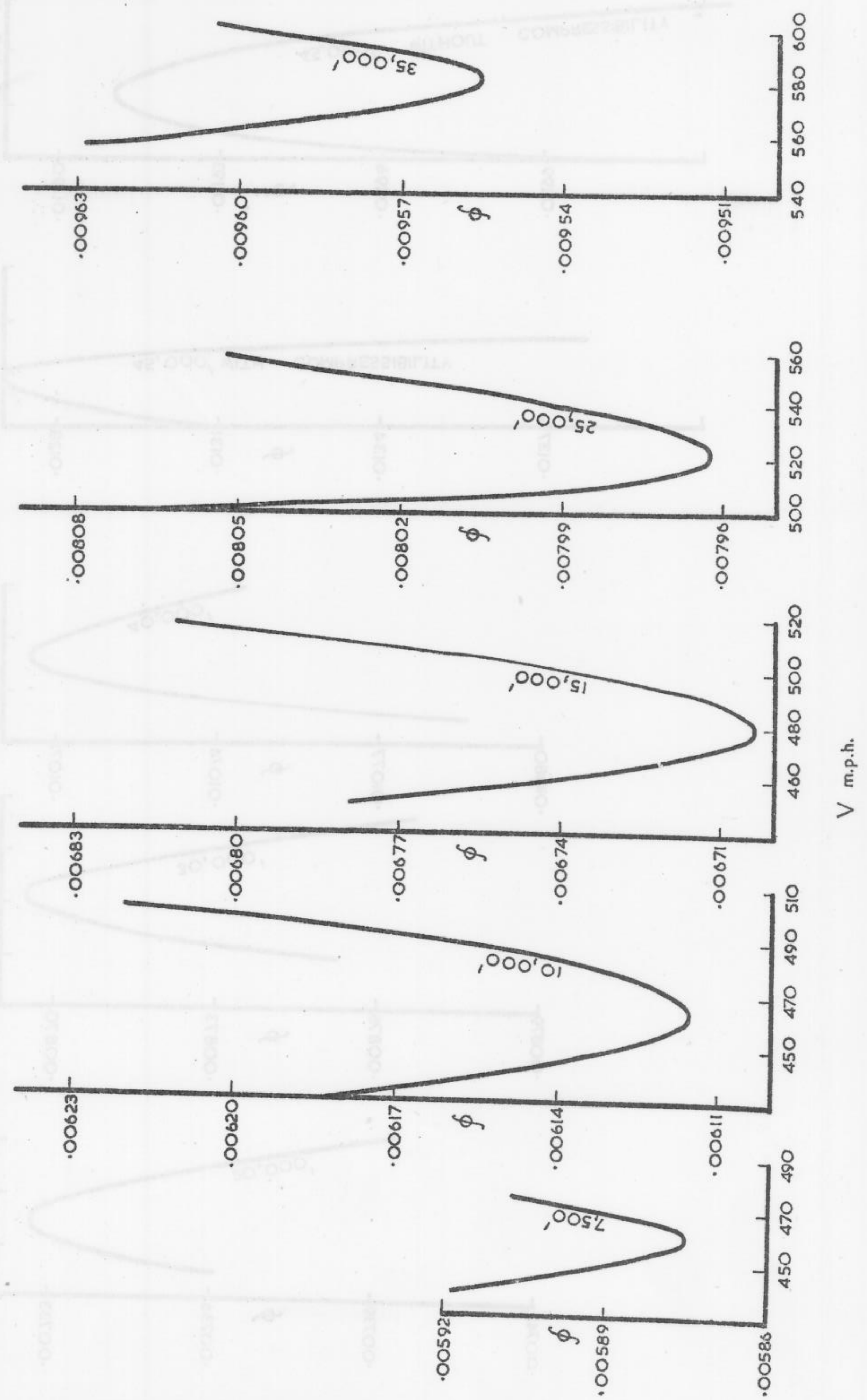
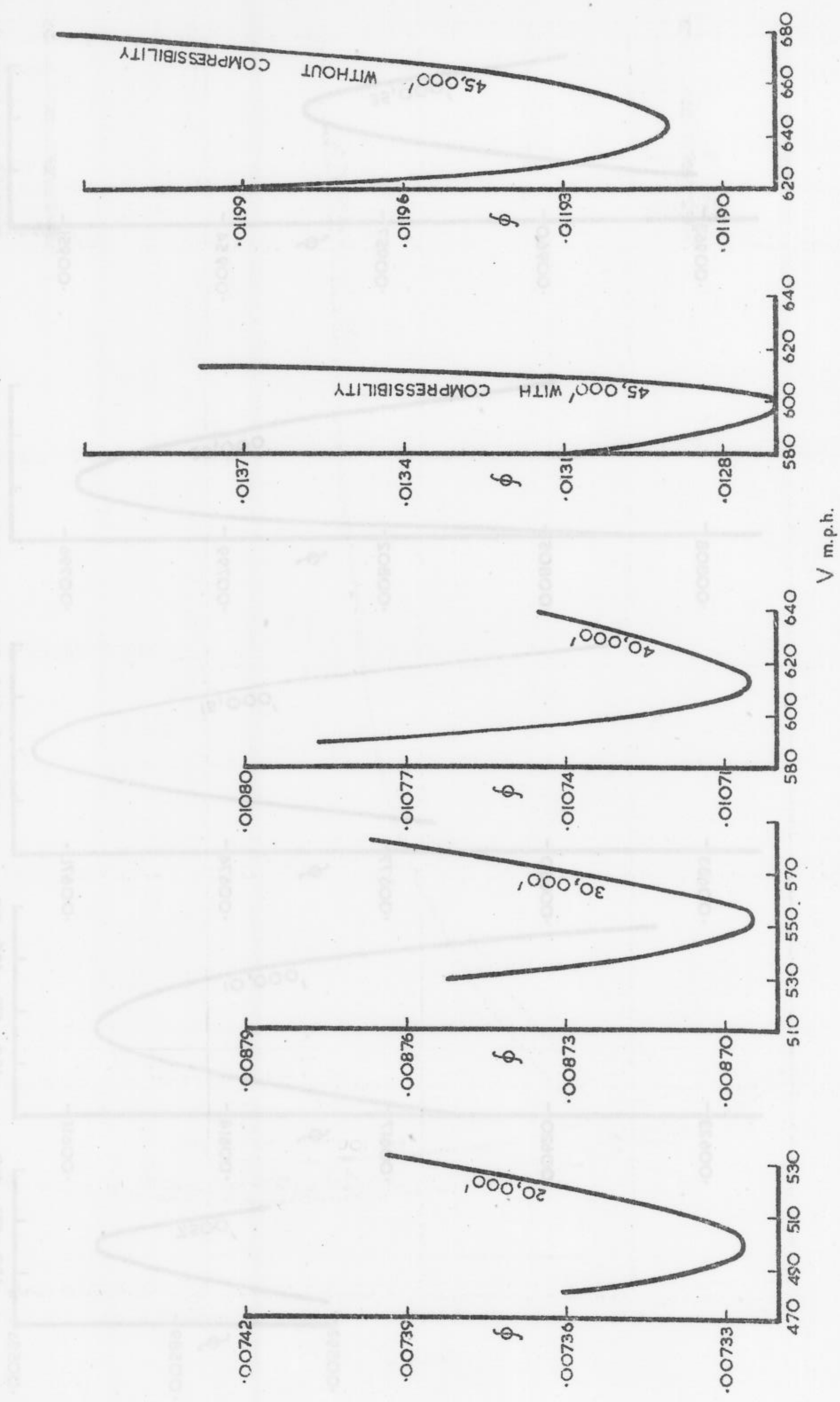
FIG. I.

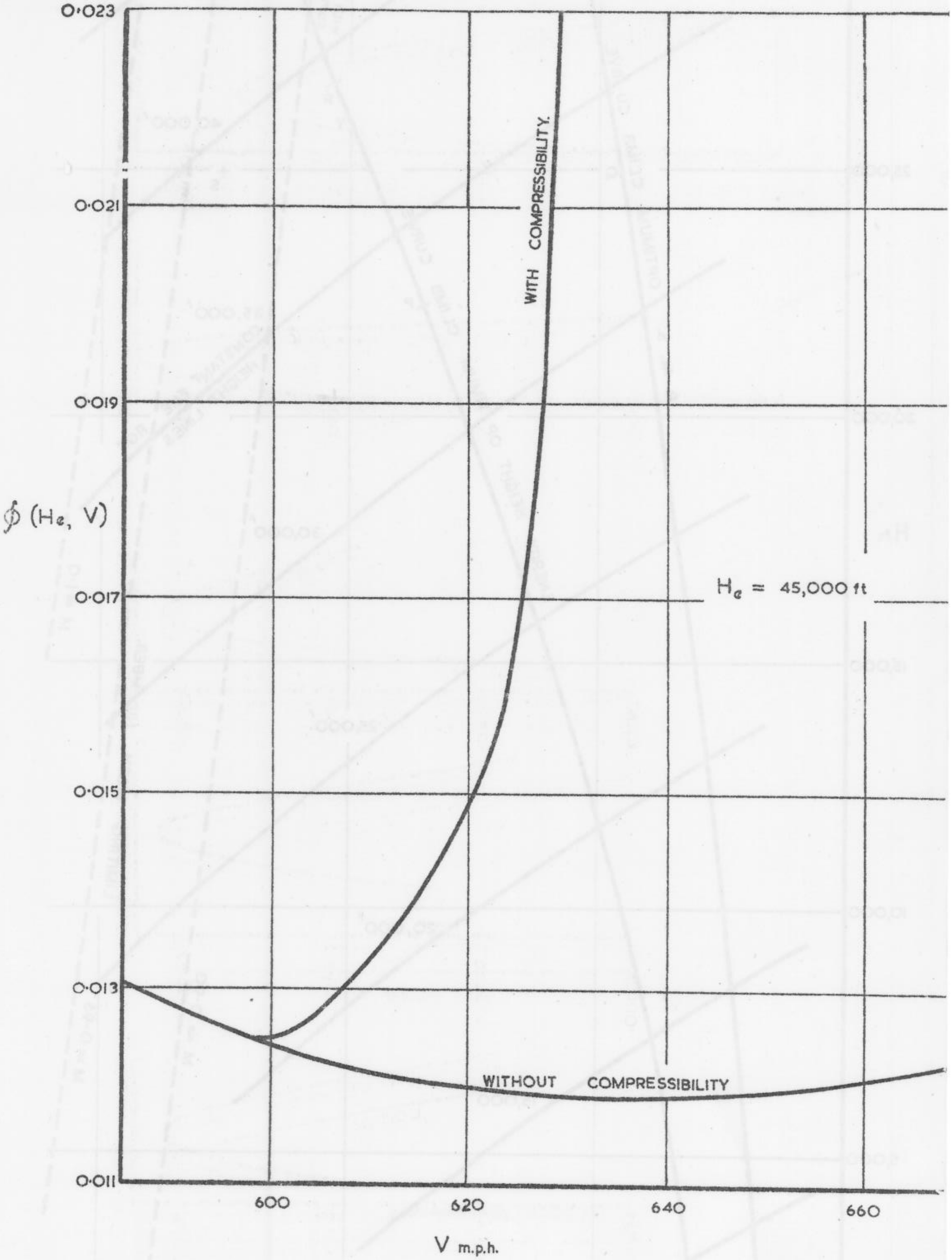


CURVES OF $\phi(H_e, V)$ AT A SERIES OF CONSTANT ENERGY HEIGHTS.

FIG. 2.

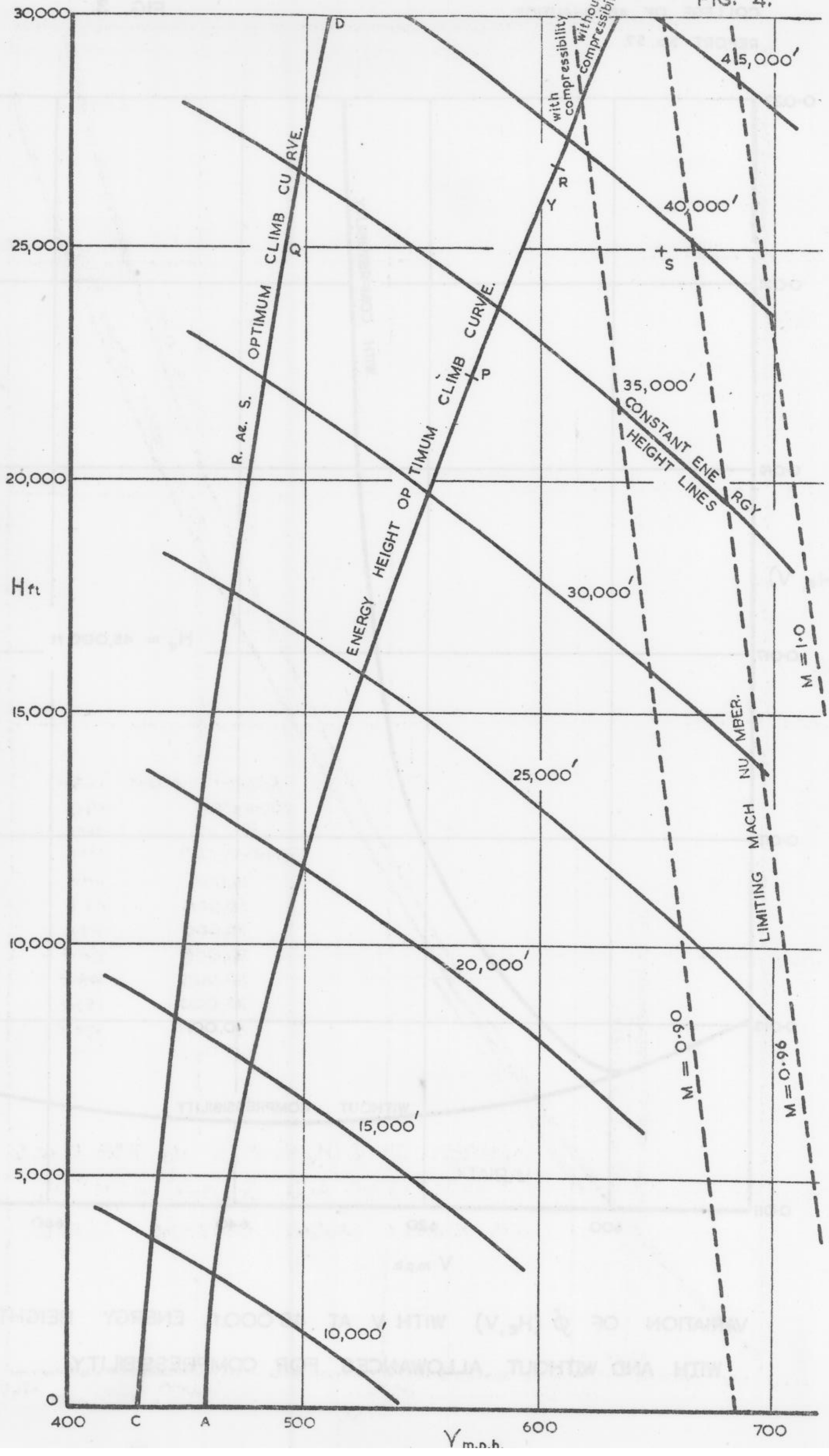


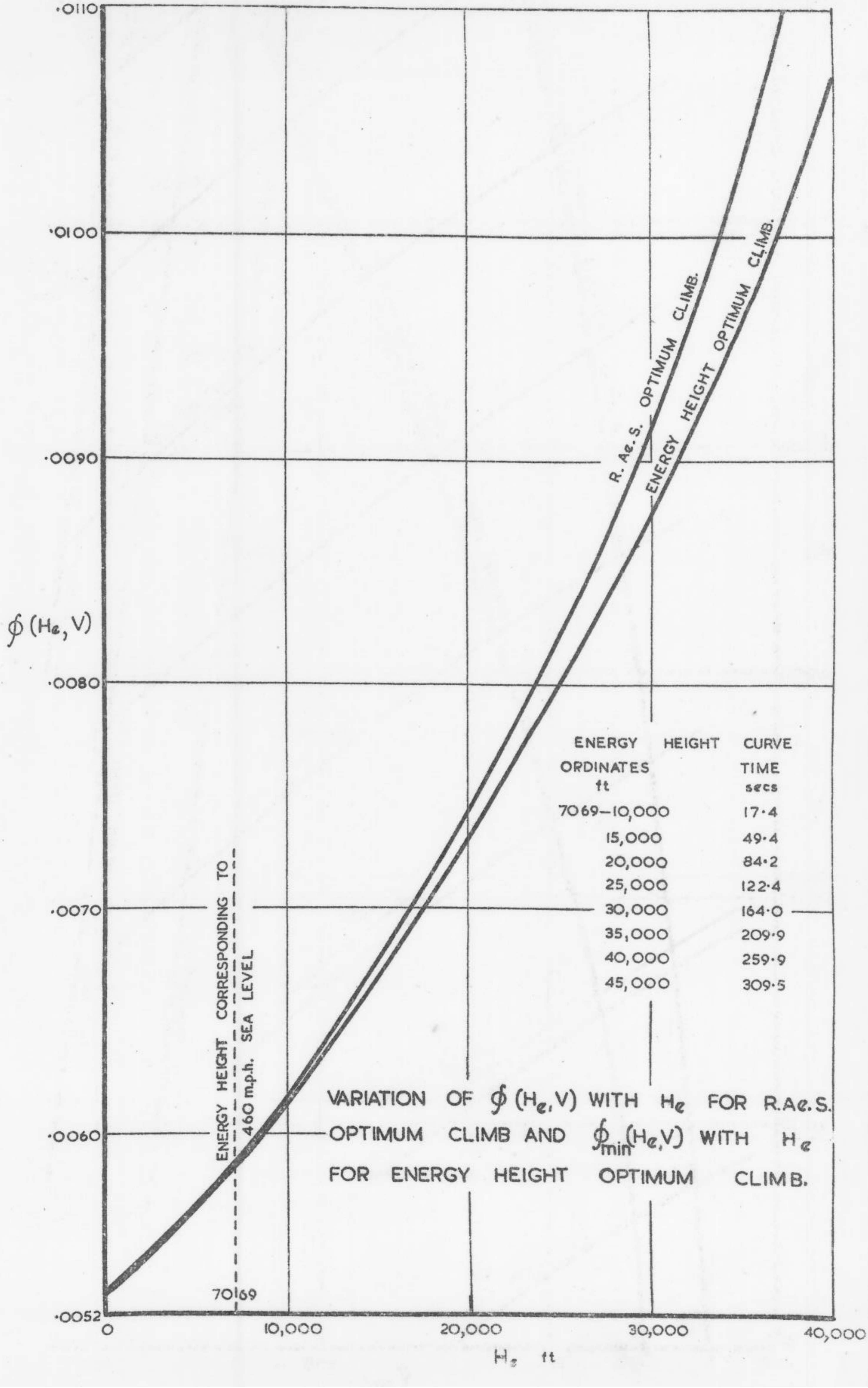
CURVES OF $\phi(H_e, V)$ AT A SERIES OF CONSTANT ENERGY HEIGHTS



VARIATION OF ϕ (H_e, V) WITH V AT 45 000 ft ENERGY HEIGHT WITH AND WITHOUT ALLOWANCES FOR COMPRESSIBILITY.

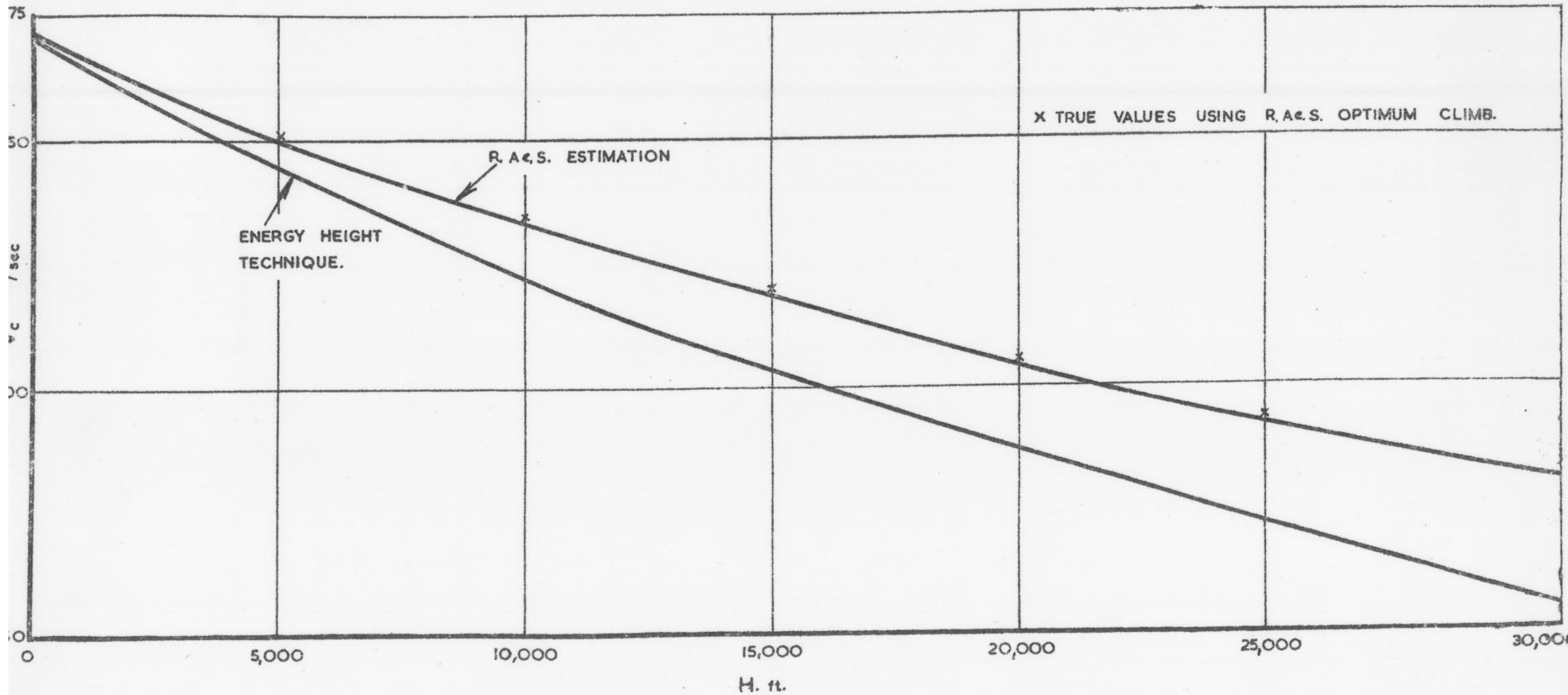
FIG. 4.


30,000
25,000
20,000
15,000
10,000
5,000
0
H_{ft}
400
C
A
500
600
700
$V_{m.p.h.}$
D
Q
R. Ae. S. OPTIMUM CLIMB CURVE.
ENERGY HEIGHT OPTIMUM CLIMB CURVE.
with compressibility
without compressibility
Y
R
P
S
CONSTANT ENERGY HEIGHT LINES
10,000′
15,000′
20,000′
25,000′
30,000′
35,000′
40,000′
45,000′
LIMITING MACH NUMBER.
M = 0·90
M = 0·96
M = 1·0



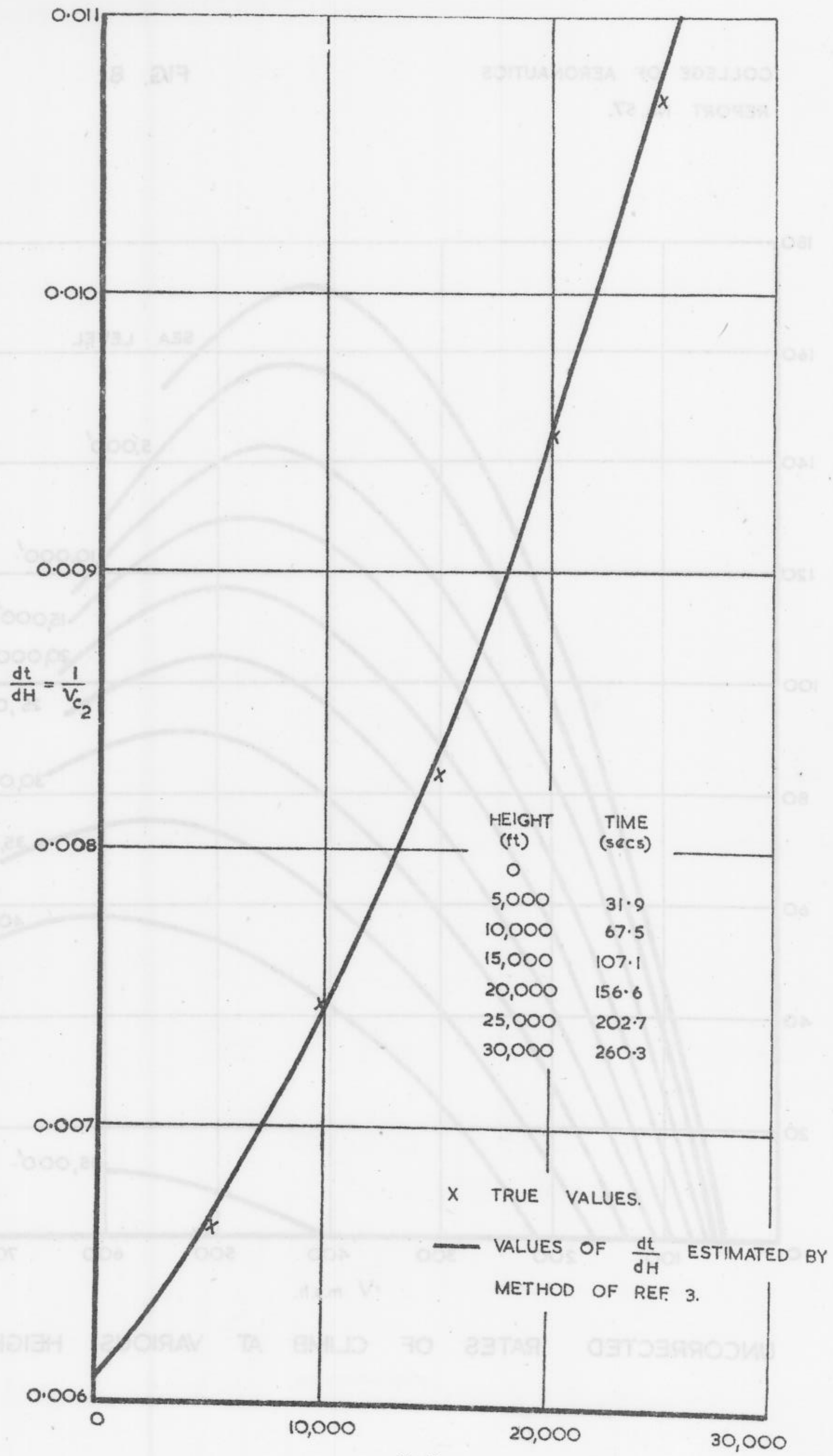
ENERGY HEIGHT ORDINATES ft	CURVE TIME secs
7069–10,000	17·4
15,000	49·4
20,000	84·2
25,000	122·4
30,000	164·0
35,000	209·9
40,000	259·9
45,000	309·5

VARIATION OF $\phi\,(H_e, V)$ WITH H_e FOR R.Ae.S. OPTIMUM CLIMB AND $\phi_{min}(H_e, V)$ WITH H_e FOR ENERGY HEIGHT OPTIMUM CLIMB.

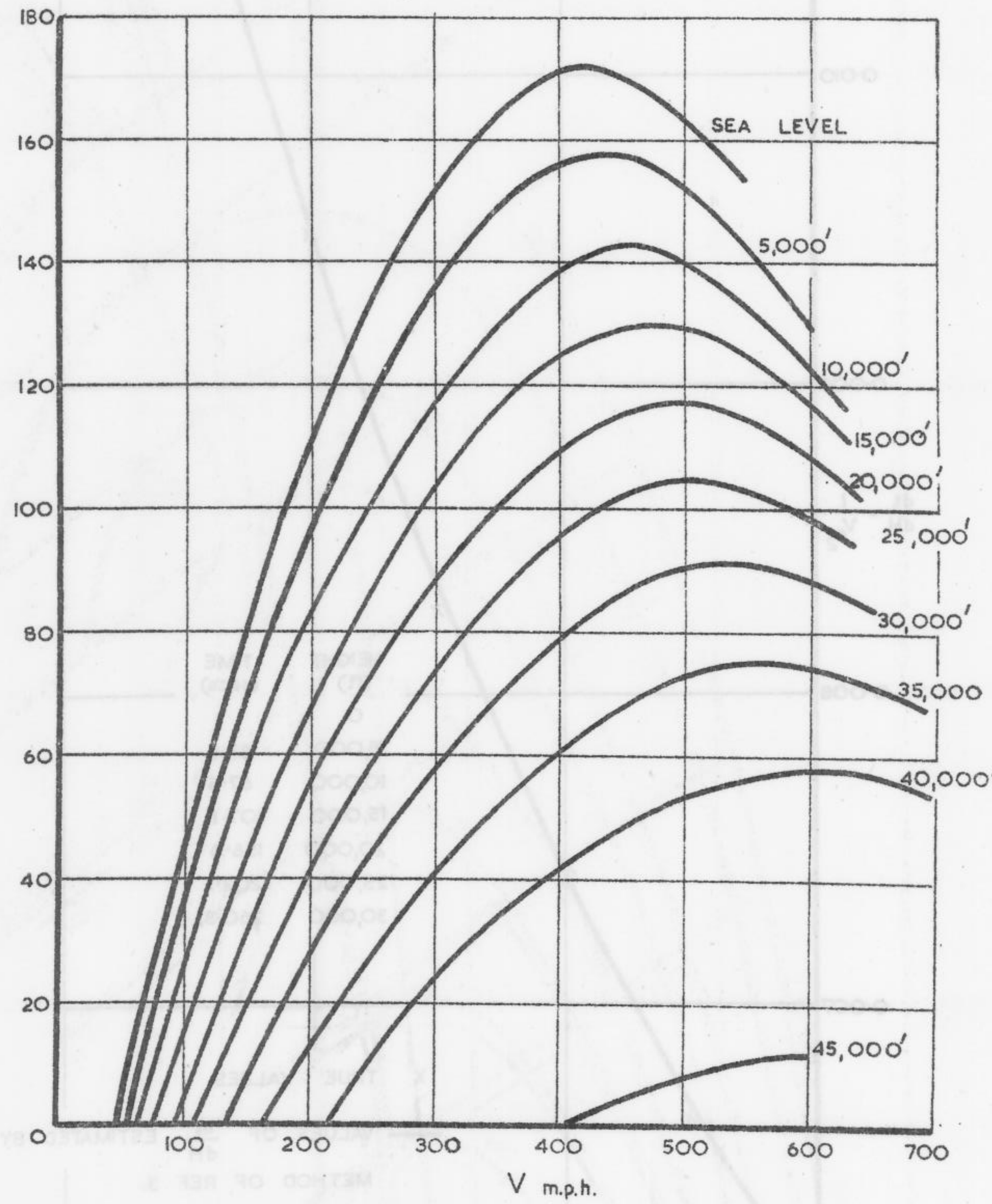


VARIATION OF RATE OF CLIMB WITH HEIGHT, R. Ae. S. AND ENERGY HEIGHT OPTIMUM CLIMBS

FIG. 7.

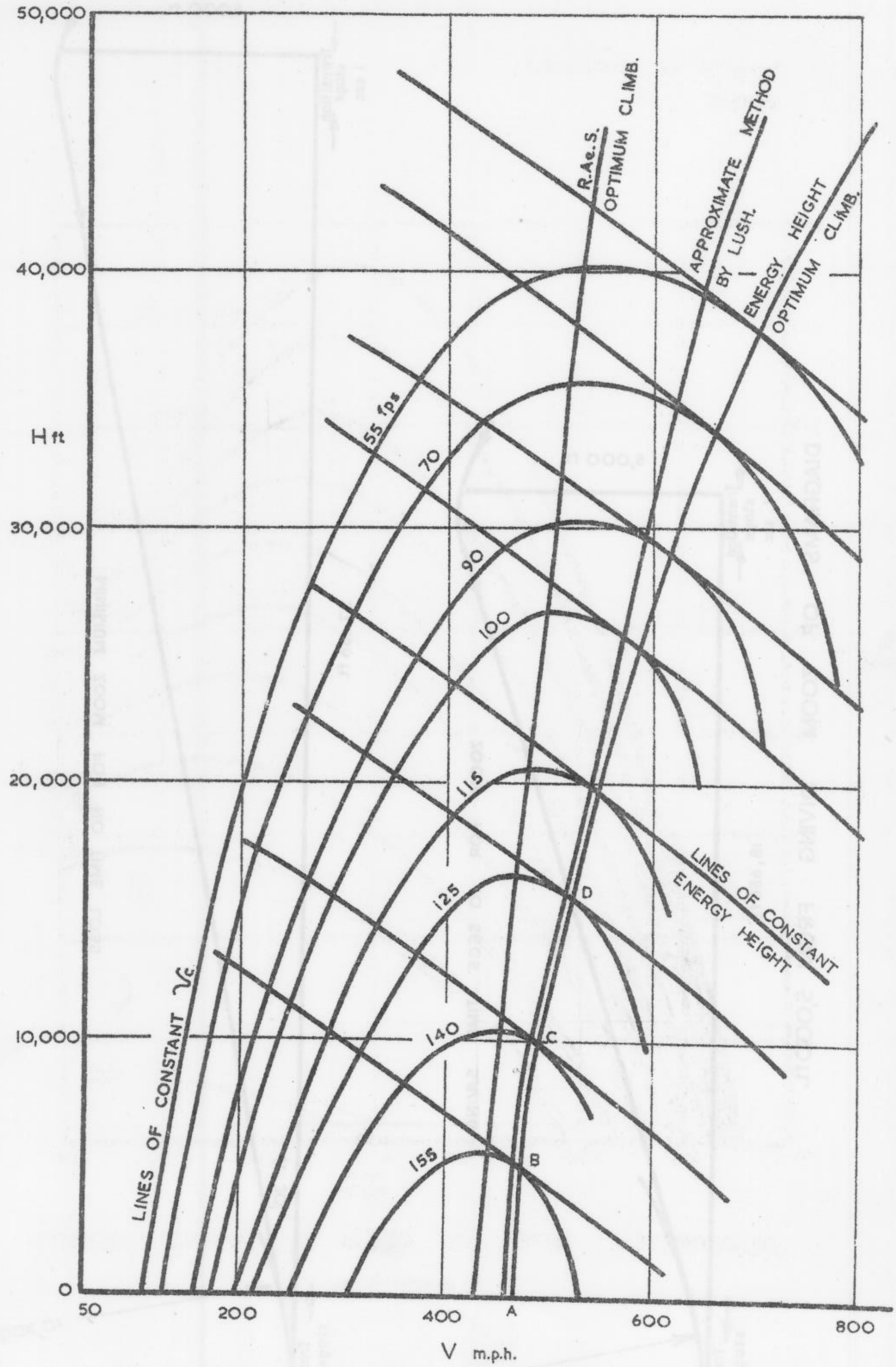


VARIATION OF $1/V_c$ WITH HEIGHT FOR TIME TO HEIGHT ESTIMATION. R. Ae. S. CLIMB.



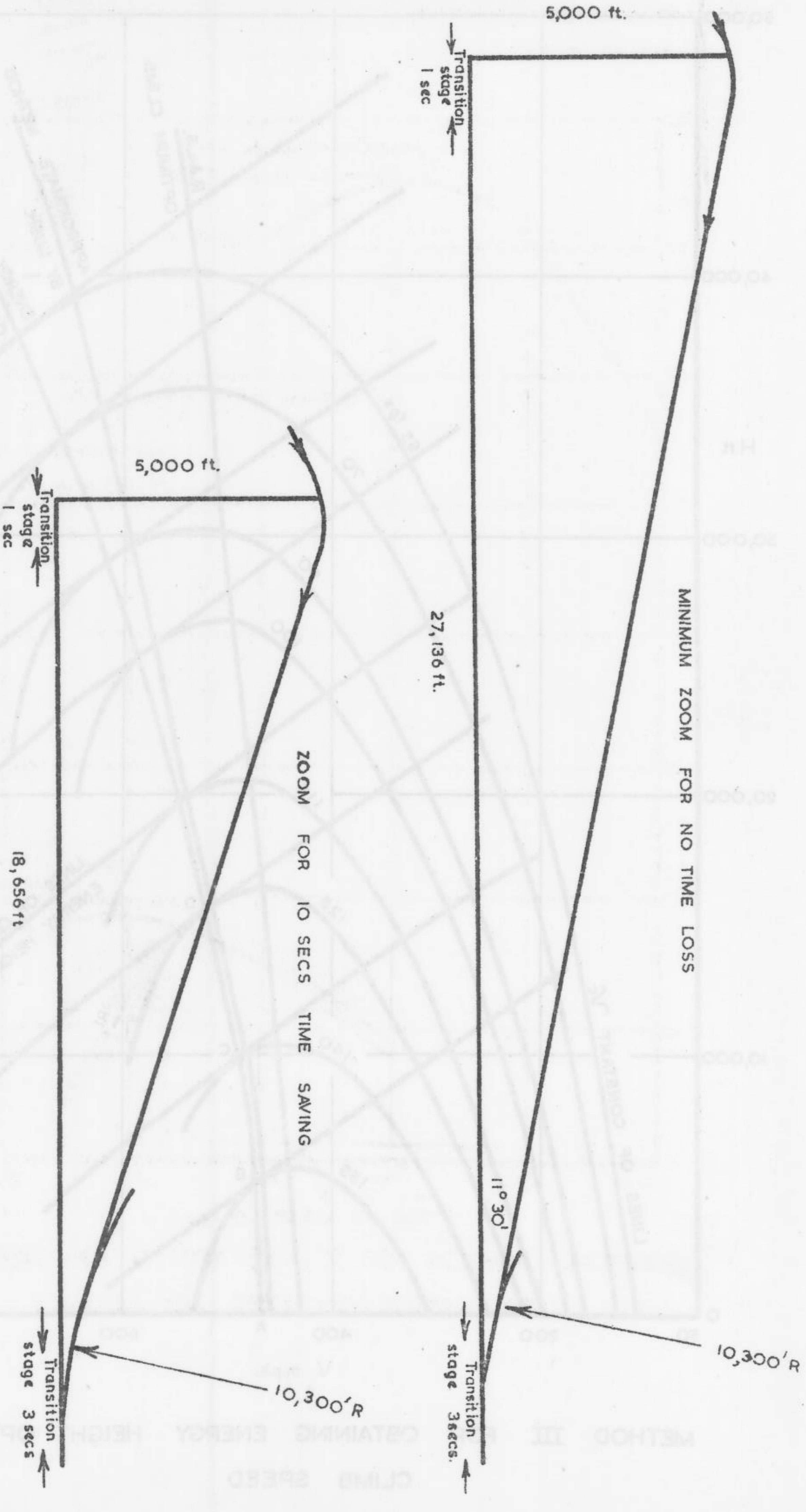
UNCORRECTED RATES OF CLIMB AT VARIOUS HEIGHTS

FIG. 9.



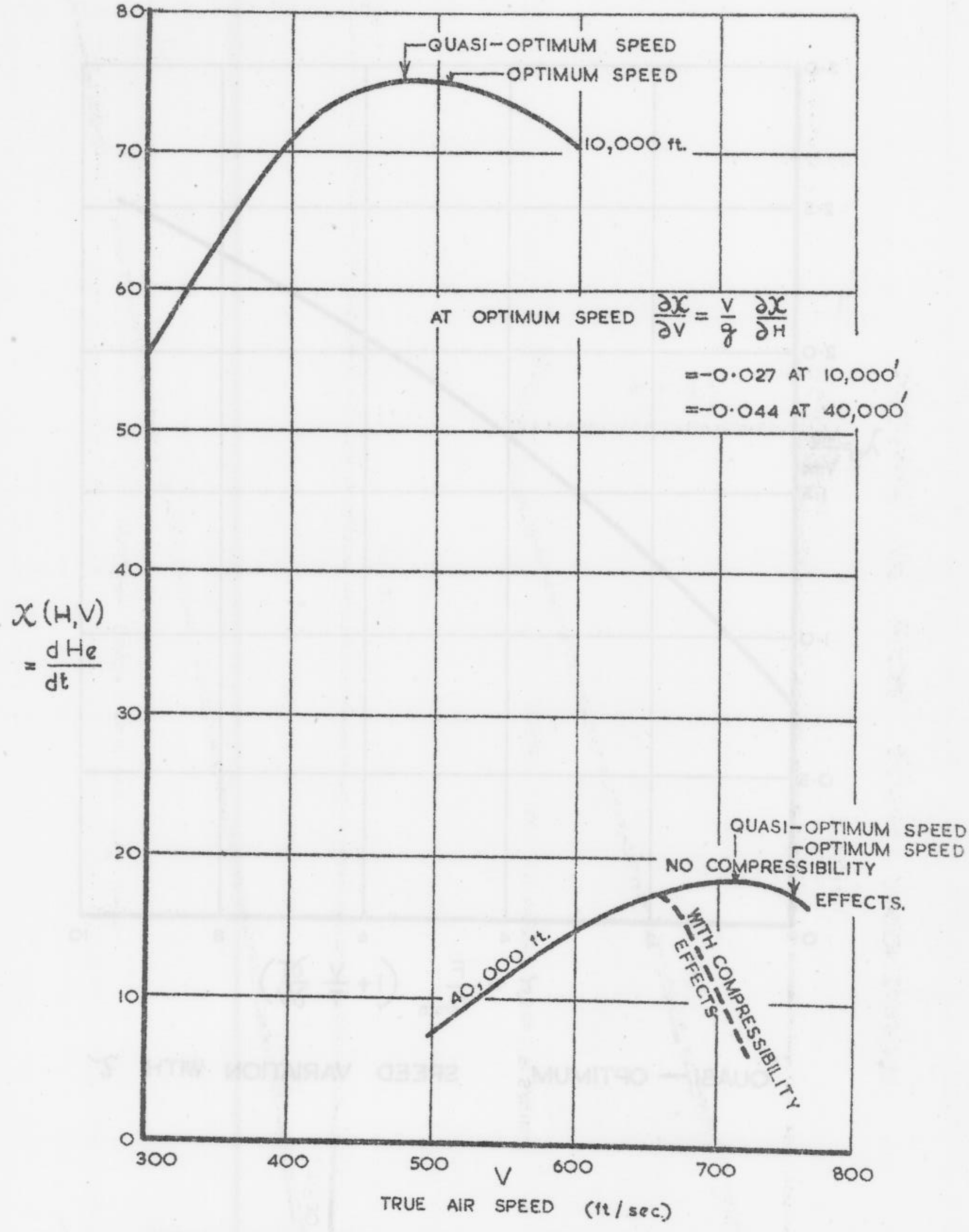
METHOD III FOR OBTAINING ENERGY HEIGHT OPTIMUM CLIMB SPEED

FIG. 10.

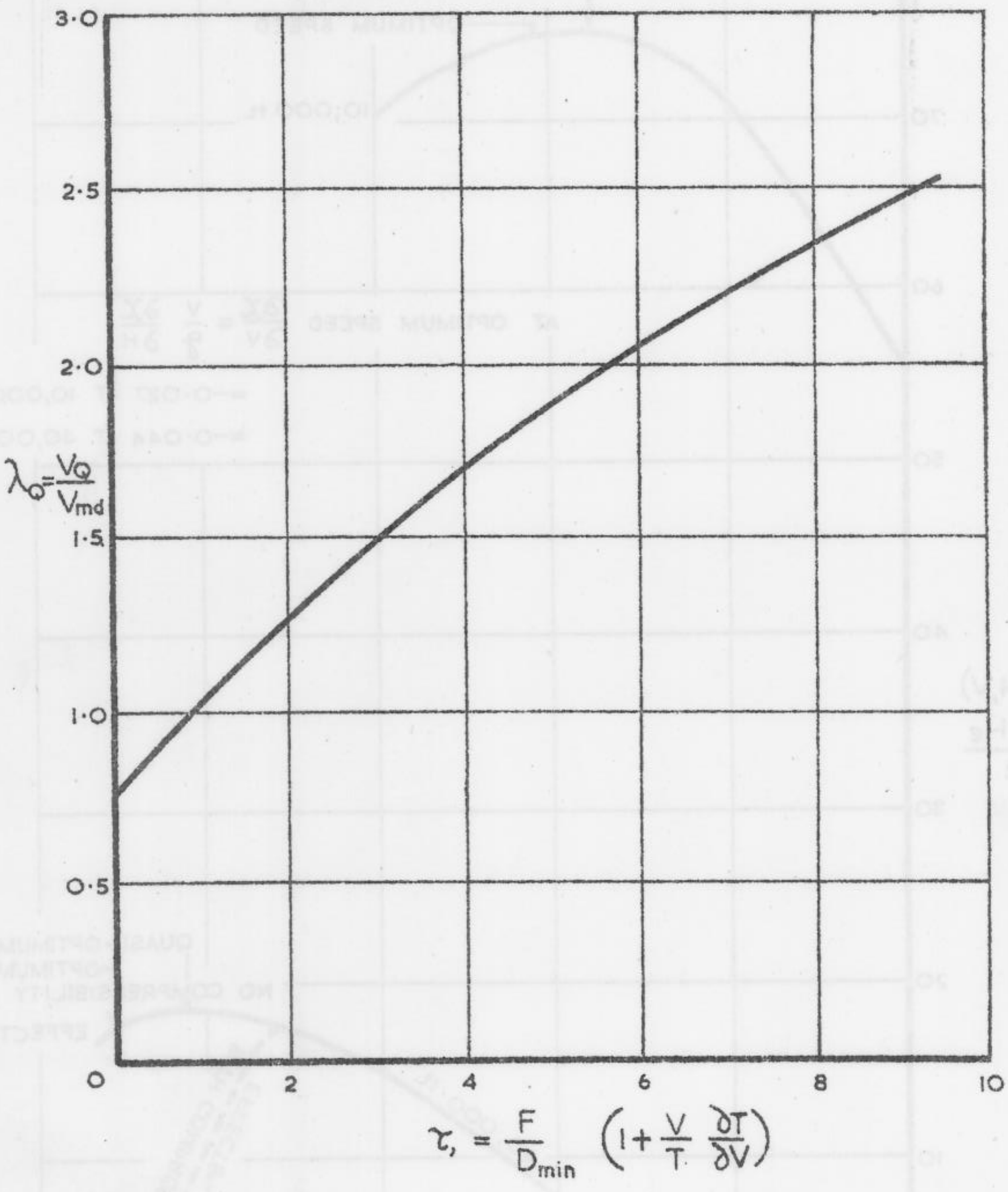


DIAGRAMS OF ZOOM DIVING FROM 5,000ft.

FIG. 11.



TYPICAL VARIATION OF χ (H,V) WITH V FOR PRESENT DAY PRODUCTION TYPE JET AIRCRAFT.



"QUASI — OPTIMUM" SPEED VARIATION WITH τ